SOLVING LINEAR SYSTEMS IN 3-D **MATH1314/GWJ/08**

The problem: solve

$a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$
$a_3x + b_3y + c_3z = d_1$

Interpreting the equations: each linear equation represents a plane in 3-space

The three intercepts are $x_{int} = d/a$
$y_{int} = d/b$
$z_{int} = d/c$

Solution cases:

three planes intersect at a single common point
(you can find an (x,y,z) triple that satisfies all 3 equation)

three planes intersect along a line in 3-space
(dependent system, infinitely-many solutions)

three planes are parallel and never intersect at all
(inconsistent system, no solution)

Algebraic methods:

(a.k.a. <u>addition</u>)

Double <u>elimination</u> – Taken by pairs <u>twice</u>, use elimination to eliminate <u>same</u> variable, resulting in two equations in 2 variables. Proceed normally from there with any convenient method.

example:

$x + y + z = 6$ eq 1
$2x - y + z = 3$ eq 2
$x + 2y – 3z = -4$ eq 3

see a "natural" in y between eq 1 and 2, and an easy single multiplier of 2 in y, eq 2 and eq 3.

$x + y + z = 6$ eq 1
$\underline{2x - y + z = 3}$ eq 2
$3x \quad +2z = 9$ eq 4

$2(2x - y + z) = 3(2)$
$x + 2y – 3z = -4$

$4x – 2y + 2z = 6$ 2 x eq 2
$\underline{x + 2y – 3z = -4}$ eq 3
$5x \quad -z \quad = 2$ or $z = 5x – 2$ eq 5

Substituting eq5 into eq 4:

$3x + 2(5x – 2) = 9$

$3x + 10x - 4 = 9$

$13x = 13$

$x = 1$

use eq 5 in x and z with the value of x to get:

$z = 5(1) – 2 = 5 – 2 = 3$

arbitrarily pick eq 1 and use values of x & z to find the value of y:

$1 + y + 3 = 6$

$y + 4 = 6$

$y = 2$

solution triple: $(1, 2, 3) = (x, y, z)$

(be sure and check these values in all 3 original equations!)

Double Substitution - Taken by pairs twice, use substitution to eliminate the same variable, resulting in two equations in 2 variables. Proceed normally from there with any convenient method.

same example:

$x + y + z = 6$	eq 1
$2x - y + z = 3$	eq 2
$x + 2y – 3z = -4$	eq 3

y is particularly easy to isolate in eq 1 and eq2:

$y = - x – z +6$	eq 4, was eq 1
$y = 2x + z – 3$	eq 5, was eq 2
$x + 2y – 3z = -4$	eq 3

substitute eq 4 into eq 3, and eq 5 into eq 3:

$x + 2(-x – z + 6) - 3z = -4$	eq 6 (3 with 4)
$x + 2(2x + z – 3) -3z = -4$	eq 7 (3 with 5)

	x -2x -2z +12 -3z = -4	simplifying eq 6
	x + 4x + 2z – 6 -3z = -4	simplifying eq 7
	-x – 5z = -16	simplified eq 6
	5x – z = 2	simplified eq 7
Solve eq 7 for z:	z = 5x – 2	and sub that into 6
	-x – 5(5x – 2) = -16	
	-x -25x +10 = -16	
	-26x = -26	
	x = 1	(as before)
use z = 5x -2:	z = 5(1) – 2 = 5 – 2 = 3	(as before)
arbitrarily pick eq 4:	y = -(1) – (3) + 6 = -4 + 6 = 2	(as before)
	solution triple (1, 2, 3)	(as before)

Matrix methods:

Row operations –

same example:

x + y + z = 6
2x - y + z = 3
x + 2y – 3z = -4

$$\begin{vmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 2 & -3 & -4 \end{vmatrix}$$

sub top from bott to get:

$$\begin{vmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 0 & 1 & -4 & -10 \end{vmatrix}$$

sub 2x top from 2^{nd} to get:

$$\begin{vmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & 1 & -4 & -10 \end{vmatrix}$$

-1/3 x 2^{nd} row to get:

$$\begin{vmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 1/3 & 3 \\ 0 & 1 & -4 & -10 \end{vmatrix}$$

sub 2nd from 3rd to get:
$$\left|\begin{array}{cccc} 1 & 1 & 1 & 6 \\ 0 & 1 & 1/3 & 3 \\ 0 & 0 & -13/3 & -13 \end{array}\right|$$

-3/13 x 3rd to get:
$$\left|\begin{array}{cccc} 1 & 1 & 1 & 6 \\ 0 & 1 & 1/3 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right|$$

sub 2nd from 1st to get:
$$\left|\begin{array}{cccc} 1 & 0 & 2/3 & 3 \\ 0 & 1 & 1/3 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right|$$

sub 2/3 x 3rd from 1st to get:
$$\left|\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1/3 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right|$$

sub 1/3 x 3rd from 2nd to get:
$$\left|\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right|$$

and read off (x, y, z) = (1, 2, 3) (same solution as before)

Taking a 3x3 determinant: (using the co-factors method)

for same example:

$$\begin{array}{l} x + y + z = 6 \\ 2x - y + z = 3 \\ x + 2y - 3z = -4 \end{array} \qquad \begin{array}{c} \begin{array}{ccc|c} x & y & z & \text{const} \end{array} \\ \left|\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 2 & -3 & -4 \end{array}\right| \end{array}$$

determinant D of the matrix of coefficients is D =
(expand along top row)

$$\det\left|\begin{array}{ccc} + & - & + \\ 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{array}\right|$$

$$= 1*\det\left|\begin{array}{cc} -1 & 1 \\ 2 & -3 \end{array}\right| -1*\det\left|\begin{array}{cc} 2 & 1 \\ 1 & -3 \end{array}\right| + 1*\det\left|\begin{array}{cc} 2 & -1 \\ 1 & 2 \end{array}\right|$$

$= 1*((-1)(-3) – (2)(1)) -1*((2)(-3) – (1)(1)) + 1*((2)(2) – (1)(-1))$

$= 1*(3-2) - 1*(-6 – 1) + 1*(4 + 1) = 1*1 – 1*(-7) + 1*5 = 1 + 7 + 5$

= 13 = D = determinant of the matrix of coefficients (by co-factors)

Cramer's Rule: (extend to 3-D, sub const column for x-column for Dx, etc)

Let Dx be determinant of coefficient matrix with the constant column substituted for the x-column

Let Dy be determinant of coefficient matrix with the constant column substituted for the y-column

Let Dz be determinant of coefficient matrix with the constant column substituted for the z-column

Let D be the determinant of the matrix of coefficients without any column substitutions

Then x = Dx/D, y = Dy/D, and z = Dz/D

for same example:

$$\begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \\ x + 2y - 3z &= -4 \end{aligned} \qquad \begin{array}{cccc} x & y & z & \text{const} \\ \hline 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 2 & -3 & -4 \end{array}$$

we already know D = 13 per example of 3-D determinants above

$$Dx = \det \begin{vmatrix} 6 & 1 & 1 \\ 3 & -1 & 1 \\ -4 & 2 & -3 \end{vmatrix} = 13$$ (details of cofactors not shown)

$$Dy = \det \begin{vmatrix} 1 & 6 & 1 \\ 2 & 3 & 1 \\ 1 & -4 & -3 \end{vmatrix} = 26$$ (details of cofactors not shown)

$$Dz = \det \begin{vmatrix} 1 & 1 & 6 \\ 2 & -1 & 3 \\ 1 & 2 & -4 \end{vmatrix} = 39$$ (details of cofactors not shown)

so x = Dx/D = 13/13 = 1 (as before)
y = Dy/D = 26/13 = 2 (as before)
z = Dz/D = 39/13 = 3 (as before)

Matrix Inversion

Use suitable calculator or computer software; take inverse of matrix of coefficients, pre-multiply it onto constants vector. Resulting vector is the solution.

for same example:

$$\begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \\ x + 2y - 3z &= -4 \end{aligned} \qquad \underbrace{\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix}}_{\text{Coeff matrix}} * \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \underbrace{\begin{vmatrix} 6 \\ 3 \\ -4 \end{vmatrix}}_{\text{consts}}$$

```
                                   inverse        consts
has the solution        | x |     | 1  1  1 |^-1 | 6 |
                        | y |  =  | 2 -1  1 | *  | 3 |
                        | z |     | 1  2 -3 |    | -4 |
```

```
calculator inverse is:            | .0769  .3846   .1538 |
(decimal approx. only)            | .5384  -.3076  .0769 |
                                  | .3846  -.0769  -.2307 |
```

```
and the matrix multiplication yields this solution vector    | 1 |
                                                             | 2 |
                                                             | 3 |
```

(same as before)

Definition: Identity Matrix I has 1's on main diagonal, 0's elsewhere

```
3x3 example:  | 1  0  0 |
              | 0  1  0 |
              | 0  0  1 |
```

Inversion by Row Operations (best done in a spreadsheet)

```
for same example:     x  +  y  +  z  = 6
                      2x -  y  +  z  = 3
                      x  + 2y – 3z = -4
```

set up augmented matrix of coefficients on the left and identity on the right

```
| 1  1  1 | 1  0  0 |
| 2 -1  1 | 0  1  0 |
| 1  2 -3 | 0  0  1 |
```

Do row ops (not shown) until identity is on the left, inverse will be on right

```
| 1  0  0 | .0769  .3846   .1538 |
| 0  1  0 | .5384  -.3076  .0769 |
| 0  0  1 | .3846  -.0769  -.2307 |
```

Use the inverse in matrix multiplication as just above to determine solution

```
[inverse] * |  6 |       | 1 |     | x |
            |  3 | =     | 2 |  =  | y |
            | -4 |       | 3 |     | z |
```

Remember – always check your solutions in all 3 original equations!!!!